How to meet low entropy LWE keys: SMAUG and TiGER

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The LWE problem

$$\begin{vmatrix} b \end{vmatrix} \equiv_q \begin{bmatrix} \\ A \\ \\ \end{bmatrix} \cdot \begin{vmatrix} s \end{vmatrix} + \begin{vmatrix} e \end{vmatrix},$$

where $A \in \mathbb{Z}_q^{m \times n}$, $s \in \mathcal{D}^n$, $e \in \mathcal{D}^m$

- Search version: Given (A, b), find s (or e)
- Decisional version: Given samples (A, b), (either LWE or uniform), decide whether they are LWE samples or uniformly random samples

LWE-based scheme is an all-rounder?

	LWE	Wish
Computing time	$\tilde{O}(n^2)$	$\tilde{O}(n)$
Known attack time	$2^{\Omega(n)}$	$2^{\Omega(n)}$

- (Pros) LWE-based scheme is secure enough
- (Cons) It is inefficient

The sparse secret LWE problem (sLWE)

$$\begin{vmatrix} b \end{vmatrix} \equiv_q \begin{bmatrix} \\ A \\ \\ \end{bmatrix} \cdot \begin{vmatrix} s \end{vmatrix} + \begin{vmatrix} e, \end{vmatrix}$$

where $A \in \mathbb{Z}_q^{m \times n}$, $s \in \mathcal{S}_h^n (H.w(s) \leq h)$, $e \in \mathcal{D}^*$

- Search version: Given (A, b), find s (or e)
- Decisional version: Given samples (A, b), (either sLWE or uniform), decide whether they are sLWE samples or uniformly random samples

Relation between sLWE and LWE; hardness of sLWE

LWE of h-dimension $\leq$ sLWE

$$\left| b \right| \equiv_q \left[\begin{array}{c} \\ A_0 \\ \\ \end{array} \right] \cdot \left| \begin{array}{c} \mathrm{s} \\ \\ \end{array} \right| + \left| e \right|,$$

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LWE of h-dimension $\leq$ sLWE

$$\begin{bmatrix} b \end{bmatrix} \equiv_q \begin{bmatrix} A_0 \end{bmatrix} \begin{bmatrix} A_1 \end{bmatrix} \cdot \begin{bmatrix} s \\ 0 \end{bmatrix} + \begin{bmatrix} e \end{bmatrix}$$

Relation between sLWE and LWE; hardness of sLWE

LWE of h-dimension $\leq$ sLWE
After permutation:

$$\begin{vmatrix} b \\ \\ \\ \\ \end{vmatrix} \equiv_q \begin{bmatrix} \\ A \\ \\ \\ \end{bmatrix} \cdot \begin{vmatrix} s \\ \\ \\ \\ \end{vmatrix} + \begin{vmatrix} e \\ \\ \\ \\ \end{vmatrix}$$

Relation between sLWE and LWE; weakness of sLWE

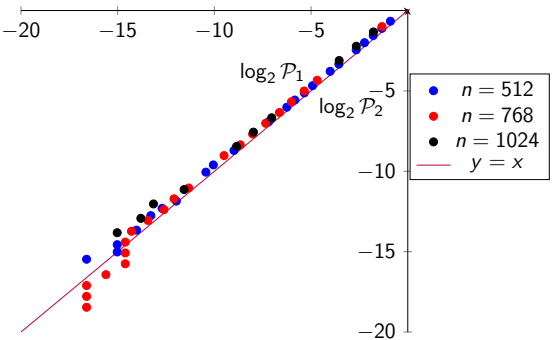
sLWE $\leq$ LWE of n-dimension : Trivial

- Lattice-based attack
  - Primal attack
  - Dual attack
- Combinatorial attack
  - MitM attack
  - BKW algorithm
- Algebraic attack
  - Arora-Ge algorithm
- Hybrid algorithm

Question: Is there an effective algorithm for sLWE?

Technical Idea: Why need more?

Main idea: $[b - A \cdot x]_q \sim \mathbb{Z}_q^m$ for $x \neq s$: $\mathcal{P}_1 = \frac{[-r,r]^m \cap \mathcal{L}}{\mathcal{L}}$, $\mathcal{P}_2 = \frac{[-r,r]^m \cap \mathbb{Z}_q^m}{\mathbb{Z}_q^m}$, $q = 3329$



Technical Idea: Why need more?

When $s \in \mathcal{S}_h^n$ and $n \sim q$, $|\mathcal{S}_h^n| = \binom{n}{h} < (q/\sigma)^h$.
It implies that an LWE sample (A, b) has a unique solution s such that

$$\begin{vmatrix} b \end{vmatrix} \equiv_q \begin{bmatrix} A \end{bmatrix} \cdot \begin{vmatrix} s \end{vmatrix} + \begin{vmatrix} e \end{vmatrix},$$

where $A \in \mathbb{Z}_q^{h \times n}$, $s \in \mathcal{D}^n$, $e \in \mathcal{D}^h$.

Desired samples with concrete parameters*

Scheme	λ	n	q	h	m
TiGER	128	512	256	128	73
	192	1024	256	84	74
	256	1024	256	198	127
SMAUG	128	512	1024	140	56
	192	768	2048	198	73
	256	1280	2048	176	85

* $\sigma = 5$

How to solve the sLWE? (Another reduction)

- Previous reduction: (n, h)-sLWE $\leq$ LWE
- New reduction: (n, h)-sLWE $\leq (n^*, h^*)$-sLWE where $n^* \leq n$ and $h^* \leq h$

Current problem:
Given $\bar{A} = (b \| A) \in \mathbb{Z}^{m \times (n+1)}$ and q, find $\bar{s}$ such that $\bar{A} \cdot \bar{s} \equiv_q e$:

$$L = \left\langle \begin{pmatrix} I_{n+1} & \\ \bar{A} & qI_h \end{pmatrix} \right\rangle \ni \begin{pmatrix} \bar{s} \\ e \end{pmatrix}$$

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Current problem:
Given $\bar{A} = (A_0 \| A_1)$ and q, find $(s_0 \| s_1)$ such that $A_0 \cdot s_0 + A_1 \cdot s_1 \equiv_q e$:

$$L = \left\langle \begin{pmatrix} I_{n-d+1} & & \\ & I_d & \\ A_0 & A_1 & qI_m \end{pmatrix} \right\rangle \ni \begin{pmatrix} s_0 \\ s_1 \\ e \end{pmatrix}$$

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Current problem:
Given $A_0 \in \mathbb{Z}^{m \times (n-d+1)}$ and B, find s_0 such that $A_0 \cdot s_0 \equiv_B s_1'$:

$$L = \left\langle \begin{pmatrix} I_{n-d+1} & \\ A_0 & B \end{pmatrix} \right\rangle \ni \begin{pmatrix} s_0 \\ s_1' \end{pmatrix}, \quad B = \begin{pmatrix} I_d & \\ A_1 & qI_m \end{pmatrix} \quad , s_1' = \begin{pmatrix} s_1 \\ e \end{pmatrix}$$

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$$L = \left\langle \begin{pmatrix} I_{n-d+1} & \\ A_0 & B \end{pmatrix} \right\rangle \ni \begin{pmatrix} s_0 \\ s_1' \end{pmatrix}, \quad B = BKZ_\beta \left(\begin{pmatrix} I_d & \\ A_1 & qI_m \end{pmatrix} \right), s_1' = \begin{pmatrix} s_1 \\ e \end{pmatrix}$$

After reduction

Scheme	λ	m^*	n	n^*	h^*
TiGER	128	497	512	340	86
	192	683	1024	704	56
	256	690	1024	683	132
SMAUG	128	585	512	292	80
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Two options for mod B

- Babai's nearest plane algorithm (BNP)
  - Polynomial-time in dimension
  - Quality depends on the size of diagonal terms
- Closest vector problem (CVP)
  - Exponential-time in dimension
  - Quality depends on $\mathrm{vol}^{1/\dim}$
- Hybrid algorithm
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How to conduct the mod B?

What is expected to get from v mod B?

$$\begin{pmatrix} 49 \\ 21 \\ 37 \end{pmatrix} \text{ mod } \begin{pmatrix} 57 & -19 & 17 \\ & 48 & 23 \\ & & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 35 \\ 8 \\ 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 15 \\ 8 \\ 4 \end{pmatrix}$$

- The last two entries are reduced by CVP
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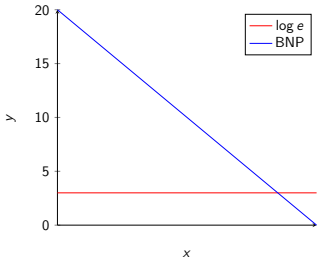
Comparison Results

$M \cdot s = e \bmod B$



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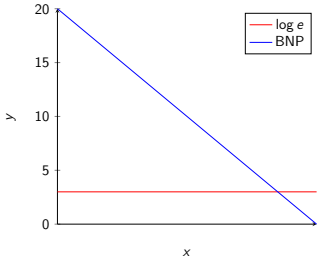

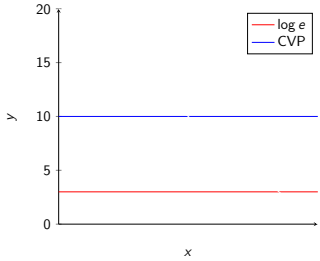

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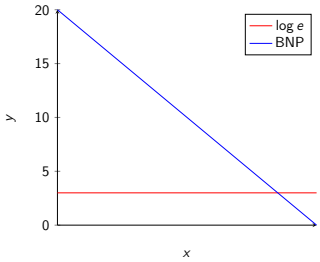

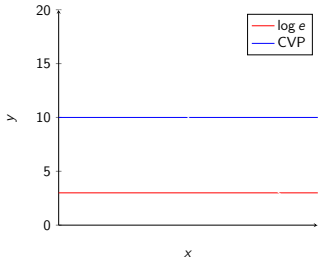

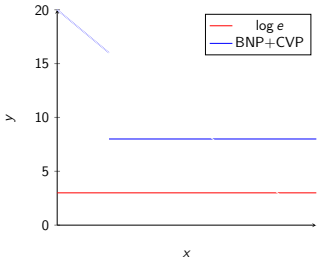

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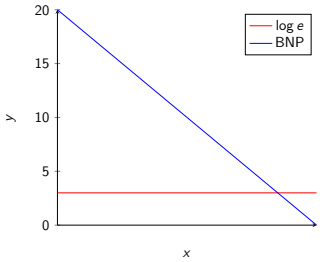

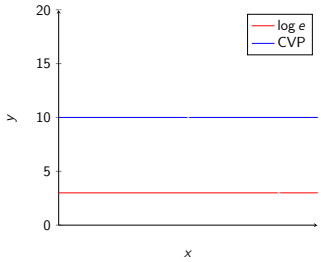

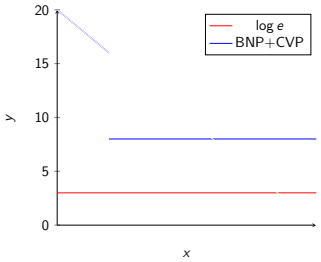

Overview for finding s

$\mathcal{L}_0$

$$\mathcal{L}_0 = \{x \in \mathbb{Z}^n \mid Hw(x) = h_0 \wedge \|M \cdot x \bmod B\|_\infty \leq \eta\}$$

$\Big|$

$$\mathcal{L}_1 = \{x \in \mathbb{Z}^n \mid HW(x) = h_1 \wedge \|\pi_r(M \cdot x \bmod B)\| \leq \eta\}$$

$\mathcal{L}_1$

$\Big|$

$$\mathcal{L}_2 \subset \{x \in \mathbb{Z}^n \mid HW(x) = h_2\}$$

$\mathcal{L}_2$

Note: $h^* = h_0 > h_1 > h_2$

Overview for finding s

SMAUG: $(n^*, h^*, m^*) = (292, 80, 585)$

$$\mathcal{L}_0 \qquad \mathcal{L}_0 = \{x \in \mathbb{Z}^{292} \mid Hw(x) = 80 \wedge \|M \cdot x \bmod B\|_\infty \leq 1.278\}$$

$$\Big|$$

$$\mathcal{L}_1 \qquad \mathcal{L}_1 = \{x \in \mathbb{Z}^{292} \mid HW(x) = 40 \wedge \|\pi_{33}(M \cdot x \bmod 8.158)\| \leq 1.278\}$$

$$\Big|$$

$$\mathcal{L}_2 \qquad \mathcal{L}_2 \subset \{x \in \mathbb{Z}^{292} \mid HW(x) = 20\}$$

Summary

- m can be chosen flexibly
- n, h can be reduced via the BKZ algorithm
- The matrix modulus B be performed as $\bmod q$ with CVPP
- To solve the (binary) SMAUG-256
  - $(n, h, q) = (512, 140, 1024) \Rightarrow (292, 80, B)$: $2^{109.5}$
  - Build the list $\mathcal{L}_2$: 2^{92}
  - Build the list $\mathcal{L}_1$: 2^{106}
  - Build the list $\mathcal{L}_0$: 2^{110}

Time complexity for binary case

Scheme	λ	m^*	n	n^*	h^*	T
TiGER	128	497	512	340	86	126
	192	683	1024	704	56	177
	256	690	1024	683	132	238
SMAUG	128	585	512	292	80	110
	192	586	768	512	132	177
	256	690	1280	683	132	225

$$\left| :) \equiv_q \left[\begin{array}{c} \textit{Question?} \\ \\ \textit{changminlee@kias.re.kr} \end{array} \right] \right. .$$

THANK YOU